Orthogonal Complements

Orthogonality: Let $x, y \in H$; $H \to$ H.S.

Then x is said to be orthogonal to y if $(x, y) = 0$ i.e. inner product of x & y is zero. We write $x \perp y$.

The relation of orthogonality is symmetric i.e. $x \perp y \Rightarrow y \perp x$

as
$$x \perp y \Rightarrow (x, y) = 0 \Rightarrow \overline{(x, y)} = \bar{0}$$
$$\Rightarrow (y, x) = 0$$
$$\Rightarrow y \perp x$$

Note If $x \perp y$ then every scalar multiple of x is orthogonal to y i.e. $\alpha x \perp y$

as $(\alpha x, y) = \alpha (x, y) = \alpha . 0 = 0$

$$\therefore x \perp y \Rightarrow \alpha x \perp y$$

Note The zero vector is the only vector which is orthogonal to itself.

Thm Pythagorean Thm

If x & y are any two orthogonal vectors in a H.S. H then

$$\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$$

Pf Since $x \perp y$, $\therefore (x, y) = 0 \Rightarrow (y, x) = 0$

Now

$$\|x+y\|^2 = (x+y, x+y) = (x, x) + (x, y) + (y, x) + (y, y)$$
$$= \|x\|^2 + 0 + 0 + \|y\|^2$$
$$= \|x\|^2 + \|y\|^2$$

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Also,

$$\|x-y\|^2 = (x-y, x-y)$$
$$= (x,x) - (x,y) - (y,x) + (y,y)$$
$$= \|x\|^2 - (x,y) - (y,x) + \|y\|^2$$
$$= \|x\|^2 - 0 - 0 + \|y\|^2$$
$$\|x-y\|^2 = \|x\|^2 + \|y\|^2$$

(Proved).

<u>Def^n</u> A vector x is said to be orthogonal to a nonempty subset S of a H.S. H if $x \perp y$ $\forall y \in S$. we write $x \perp S$.

Two non-empty subsets S_1 & S_2 of a H.S. H are said to be orthogonal if $x \perp y$, $\forall x \in S_1$, $\forall y \in S_2$

we write $S_1 \perp S_2$.

<u>Orthogonal Complement</u>

Let S be a non-empty subset of a H.S. H. The orthogonal complement of S, written as $S^\perp$ is defined by—

$$S^\perp = \{x \in H \mid x \perp y \ \forall y \in S\}$$

i.e. set of those vectors of H, which are orthogonal to every vector in S.

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Thm Let S be a non-empty subset of a H.S. H. Then $S^\perp$ is a closed linear subspace of H.

Pf:- We have →

$$S^\perp = \{x \in H \mid (x, y) = 0 \ \forall \ y \in S\}$$

Since $(0, y) = 0 \ \forall \ y \in S$, $\therefore \ 0 \in S^\perp$

$\Rightarrow S^\perp$ is non-empty.

First we shall show that $S^\perp$ is a subspace of H and then shall show that $S^\perp$ is closed.

$S^\perp$ is a subspace of H –

Let $x_1, x_2 \in S^\perp$ and α, β be any scalars.

Then $(x_1, y) = 0$ & $(x_2, y) = 0 \ \forall \ y \in S$.

$\forall \ y \in S$, we have →

$$(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$$
$$= \alpha . 0 + \beta . 0 = 0$$

$$\Rightarrow \alpha x_1 + \beta x_2 \in S^\perp$$

$\therefore$ $S^\perp$ is a subspace of H.

$S^\perp$ is closed –

We shall prove that if x is any limit point of $S^\perp$ then $x \in S^\perp$ i.e. $S^\perp$ contains all its limit points. Then $S^\perp$ will be closed.

Let x be any limit point of $S^\perp$. Then $\exists$ a seqn $\langle x_n \rangle$ in $S^\perp$ s.t.

$x_n \to x$

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Let $y \in S$. Since $x_n \in S^\perp$ $\forall n$,

$\therefore \quad (x_n, y) = 0 \quad \forall n$

$\Rightarrow \lim (x_n, y) = 0$

$\Rightarrow (\lim x_n, y) = 0$, $\because$ inner product is continuous

$\Rightarrow (x, y) = 0 \quad \forall y \in S$

$\Rightarrow x \in S^\perp$

$\Rightarrow S^\perp$ is closed

(Proved)

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